Investigating the extremal martingale measures with pre-specified marginals

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Financial motivation

- Financial context: $(S_i)_{i=0,1,2}$ an asset price s.t. $S_0 = 1$, $S_1 = X$ and $S_2 = Y$.
- All European options prices, with maturities 1 and 2, are given.
  $\Rightarrow$ marginals μ, ν at time 1 and 2 are given.
- No-arbitrage condition $\Rightarrow (S_i)_{i=0,1,2}$ is a martingale.

We introduce the set:

$$\mathcal{M}(\mu, \nu) := \{\mathbb{P} : \ X \ \mu, Y \ \nu, \mathbb{E}^{\mathbb{P}}[Y|X] = X\}.$$

$\mathcal{M}(\mu, \nu)$ is a convex set.

Set $\mathcal{M}(\mu, \nu)$

- [Strassen(1965)] Theorem: $\mathcal{M}(\mu, \nu)$ is not empty if and only if

$\mu \preceq \nu$ in the sense of convex ordering.

- Convex ordering: $\mu \preceq \nu$ iff

$$\int f d\mu \leq \int f d\nu \text{ for all convex functions } f$$

In particular μ and ν have the same mean:

$$\int x\mu(dx) = \int y\nu(dy)$$

Primal problem

- Sup-problem:

$$\overline{P}(\mu,\nu,f) = \sup_{Q\in\mathcal{M}(\mu,\nu)} \mathbb{E}^Q[f(X,Y)].$$

- Inf-problem:

$$\underline{P}(\mu,\nu,f) = \inf_{Q\in\mathcal{M}(\mu,\nu)} \mathbb{E}^Q[f(X,Y)].$$

Dual problem

Dual formulation of the inf and sup-problems

- Super-hedging value

$$\overline{D}(\mu,\nu,f)=\inf_{(\varphi,\psi,h)\in\overline{\mathcal{H}}}\int\varphi(x)\mu(dx)+\int\psi(y)\mu(dy),$$

- Sub-hedging value

$$\underline{D}(\mu,\nu,f)=\sup_{(\varphi,\psi,h)\in\underline{\mathcal{H}}}\int\varphi(x)\mu(dx)+\int\psi(y)\mu(dy),$$

with

$$\overline{\mathcal{H}} = \Big\{(\varphi,\psi,h)\text{ s.t. }\varphi(x)+\psi(y)+h(x)(y-x)\geq f(x,y)\Big\},$$

$$\underline{\mathcal{H}} = \Big\{(\varphi,\psi,h)\text{ s.t. }\varphi(x)+\psi(y)+h(x)(y-x)\leq f(x,y)\Big\}.$$

Financial interpretation of the dual problem

The super-hedging value $\overline{D}(\mu, \nu, f)$ is the cost of the cheapest super-hedging strategy of the derivative $f(X, Y)$ by

- Static trading on the European options with maturities 1 and 2, represented by (φ, ψ)
- Dynamic trading on the underlying asset S, represented by h

Cheapest super-hedging because:

- Cheapest initial cost: $\inf \int \varphi(x)\mu(dx) + \int \psi(y)\mu(dy)$
- Super-hedging: $\varphi(x) + \psi(y) + h(x)(y - x) \geq f(x, y)$

[Beiglboeck(2013)]

No duality gap:

If f is upper semi-continuous with linear growth, then there is no duality gap, i.e.

$$\sup_{Q \in \mathcal{M}(\mu,\nu)} \mathbb{E}^{Q}[f(X,Y)] = \inf_{(\varphi,\psi,h)\in\overline{\mathcal{H}}} \mu(\varphi) + \nu(\psi)$$

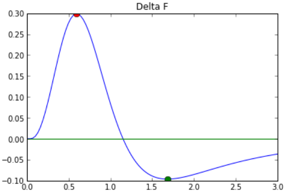
Moreover, the supremum is attained, i.e. there exists a maximizing martingale measure.

$$\exists \mathbb{P}_{\star}, \quad \sup_{Q \in \mathcal{M}(\mu,\nu)} \mathbb{E}^{Q}[f(X,Y)] = \mathbb{E}^{\mathbb{P}_{\star}}[f(X,Y)]$$

Hypotheses.

1. μ, ν have positive densities p_μ, p_ν such that $\mu \preceq \nu$ and $\int_0^\infty x p_\mu(x) = \int_0^\infty x p_\nu(x) = 1$.
2. Denote $\delta F = F_\nu - F_\mu$. Suppose that δF has a SINGLE LOCAL MAXIMIZER m.

Similarly: $G_\mu(x) = \int_0^x y\mu(dy)$, $G_\nu(x) = \int_0^x y\nu(dy)$, $\delta G = G_\nu - G_\mu$.



[Hobson and Klimmek(2013)]

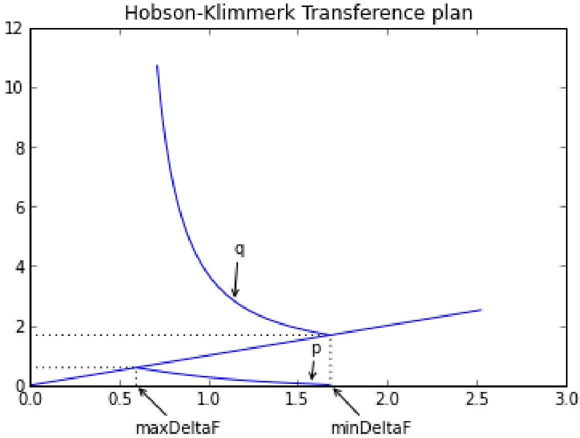
- Derive explicit expressions for the coupling giving a model-free sub-replicating price of a at-the-money forward start straddle of type II C^1_{II}:

$$C^1_{II}(x, y) = |y - x|, \quad \forall x, y > 0,$$

- The optimal martingale transport is concentrated on a three point transition graph $\{p(x), x, q(x)\}$ where p and q are two decreasing functions.

$$\mathbb{P}_\star(Y \in \{p(X), X, q(X)\}) = 1$$

[Hobson and Klimmek(2013)]



[Beiglböck and Juillet(2012)]

- Introduce the concept of *left-monotone* and *right-monotone* transference plans and prove its existence and uniqueness.
- Show that these transference plan realise the optimum in the martingale optimal transport problem, for a certain class of payoffs:
  - $f(x, y) = h(x - y)$ where h is a differentiable function whose derivative is strictly convex.
  - $f(x, y) = \Psi(x)\phi(y)$ where Ψ is a non-negative decreasing function and ϕ a non-negative strictly concave function.
- Existence result only: no explicit characterization of the optimal measure.

[Henry-Labordère and Touzi(2013)]

- Extend the results of [Beiglböck and Juillet(2012)] to a wider set of payoffs:

$$f_{xyy} > 0$$

  This set contains the coupling treated in [Beiglböck and Juillet(2012)] ($f(x,y) = h(x-y)$ and $f(x,y) = \Psi(x)\phi(y)$).
- Give explicit construction of the optimal measure, which are of left-monotone transference plan type.

Definition

Basic left-monotone transference plan $(x_\star, L_d, L_u)$, where $x_\star \in \mathbb{R}_+^*$ and L_d, L_u are positive continuous functions on $]0, \infty[$:

i) $L_d(x) = L_u(x) = x$, for $x \leq x_\star$;

ii) $L_d(x) < x < L_u(x)$, for $x > x_\star$;

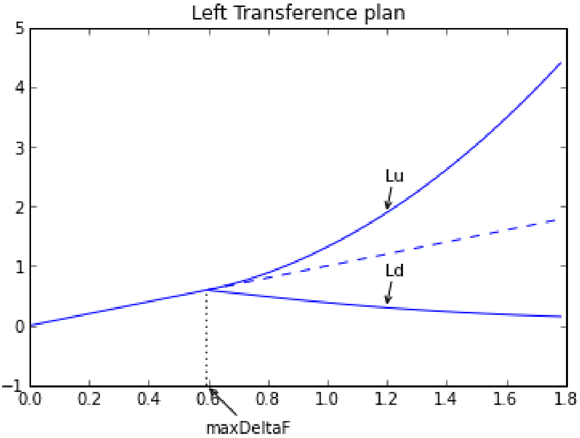
iii) on the interval $]x_\star, \infty[$, L_d is decreasing, L_u is increasing;

iv) $\mathcal{L}\mu = \nu$ where the transition kernel $\mathcal{L}$ is defined by

$$\mathcal{L}(x, dy) = \delta_x \mathbb{1}_{x \leq x_\star} + (q(x)\delta_{L_u(x)} + (1 - q(x))\delta_{L_d(x)})\mathbb{1}_{x > x_\star}$$

where $q_L(x) := \frac{x - L_d(x)}{L_u(x) - L_d(x)}$.

Basic left-monotone transference plan $(x_\star, L_d, L_u)$



Basic right monotone transference plan

Basic right-monotone transference plan $(x^\star, R_d, R_u)$, where $x_\star \in \mathbb{R}_+^*$ and R_d, R_u are positive continuous functions on $]0, \infty[$:

i) $R_d(x) = R_u(x) = x$, for $x \geq x_\star$;

ii) $R_d(x) < x < R_u(x)$, for $x < x_\star$;

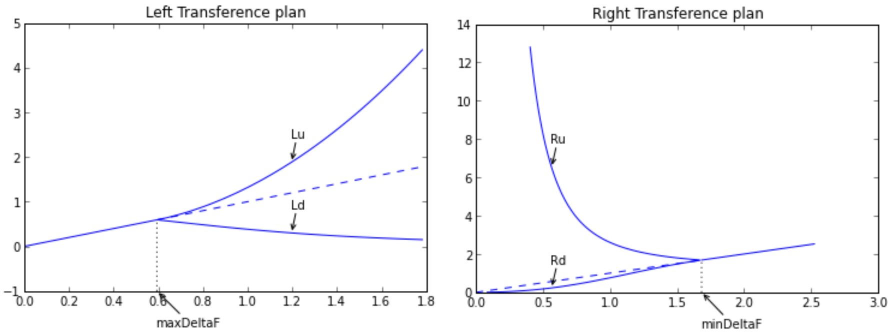
iii) On the interval $]0, x_\star[$, R_d is increasing, R_u is decreasing,

iv) $\mathcal{L}\mu = \nu$ where the transition kernel $\mathcal{L}$ is defined by

$$\mathcal{L}(x, dy) = \delta_x \mathbb{1}_{x \leq x\star} + (q(x)\delta_{R_u(x)} + (1 - q(x))\delta_{R_d(x)})\mathbb{1}_{x > x\star}$$

where $q_L(x) := \frac{x - R_d(x)}{R_u(x) - R_d(x)}$.

Basic right monotone transference plan



[Hobson and Klimmek(2013)] transference plan $\mathbb{Q}_{HK}(\mu,\nu)$

Type II forward start option: $C(X,Y) = |Y - X|$.

[Hobson and Klimmek(2013)] prove that

$$\inf_{Q \in \mathcal{M}(\mu,\nu)} \mathbb{E}^{Q}\left[|Y - X|\right] = \mathbb{E}^{\mathbb{Q}_{HK}(\mu,\nu)}\left[|Y - X|\right]$$

- The measure $\mathbb{Q}_{HK}(\mu,\nu)$ is an extremal point of $\mathcal{M}(\mu,\nu)$ (by considering the support and the construction of $\mathbb{Q}_{HK}(\mu,\nu)$).

F-Increasing transference plan (Laachir I., 2014)

A pair of functions (l, m) is a F-increasing transference plan if the following conditions are fulfilled

1. l and m are increasing.
2. $l(x) < x < m(x)$ for all $x > 0$.
3. $l(0) = 0$, $\lim_\infty l(x) = z_F^\star$ and $m(0) = z_F^\star$ (zero of δF).
4. $\mathcal{L}\mu = \nu$, where the transition kernel $\mathcal{L}$ is defined by $\mathcal{L}(x, dy) = q(x)\delta_{l(x)} + (1 - q(x))\delta_{m(x)}$ where $q_L(x) := \frac{m(x)-x}{m(x)-l(x)}$.

Rmk: $z_F^\star$ zero of the function $\delta F := F_\nu - F_\mu$.

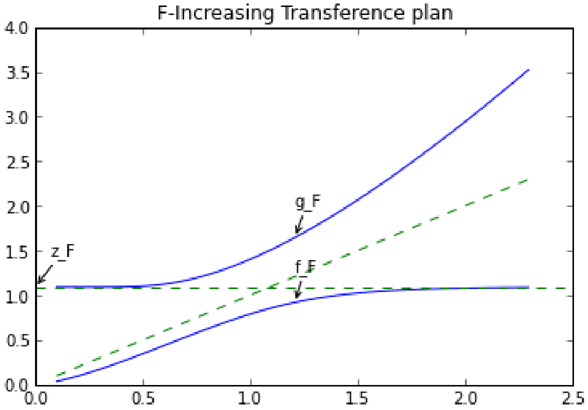
Existence

Proposition

The F-increasing transference (l, m) exists and it is unique. For every $x > 0$, $(l(x), m(x))$ is the unique solution of the system of equations

$$\begin{aligned} F_\nu(m(x)) + F_\nu(l(x)) - F_\nu(z_F^\star) &= F_\mu(x) \\ G_\nu(m(x)) + G_\nu(l(x)) - G_\nu(z_F^\star) &= G_\mu(x) \end{aligned}$$

Illustration



Extremal Points

- Very few explicit transference plans are known
- They are all extremal points of $M(\mu, \nu)$ (consider the support for 2 points plans) and share a common structure
- The convex set $M(\mu, \nu)$ is weakly compact and metrizable. By the Choquet representation theorem, any $Q \in M(\mu, \nu)$ satisfies

$$Q = \int Q_\alpha d\mu(\alpha)$$

for some probability measure on the extremal points Q_α.

(e.g.: the Black-Scholes case)

Douglas and the WEP

Theorem

$Q \in M(\mu, \nu)$ is extremal if and only if the set

$$\left\{\varphi(x) - \psi(y) + h(x)(y-x) \backslash (\varphi, \psi, h) \in L^1(\mu) \times L^1(\nu) \times L^1(x\mu)\right\}$$

is dense in $L^1(Q)$.

Definition (WEP)

$Q \in M(\mu, \nu)$ has the Weak Exact PRP iff

$$\forall f \in L^1(Q), \exists (\varphi, \psi, h) \text{ s.t. } f(x,y) = \varphi(x) - \psi(y) + h(x)(y-x) \ a.s.$$

Some consequences of Douglas theorem

Proposition

Q is extremal in $M(\mu, \nu)$ iff for any
$Q' \in M(\mu, \nu)$ $Q' << Q \implies Q' = Q$.

WEP and the Poisson Equation, 1

WEP is certainly a very strong property. As an illustration, consider the case where for every x, $x \in suppQ(x,.)$. If f is such that $f(x,x) = 0$, y setting $x = y$ we get that $\phi = \psi$. Then:

Proposition

- *ψ solves the Poisson Equation $(I - Q(x,.))\psi = v$ where $v(x) = Q(x,.)f(x.)(x)$*
- *the potential kernel $G(x,.)$ applied to v is finite, and $\psi(x) = G(x,.)v(x) + Q(x,.)^{\infty}\psi(x)$ where $Q(x,.)^{\infty}\psi$ is a $Q(x,.)$ invariant function.*

WEP and the Poisson Equation, 2

In case $Q(x,.) \in M(\mu,\nu)$ has 3 point support with $x \in Q(x,.)$, let $Q^*(x,.)$ the CRR kernel supported on $Q(x, R_+ \backslash \{x\})$.

Proposition

If for any bounded f with $f(x,x) = 0$, the PE associated to Q^* has a solution with linear growth, then Q has the WEP.

Let ψ such that $(I - Q^*)\psi = Q^* f(x,.)(x)$. Since $Q^*(x,.)$ has 2 points support, $f(x,y) + \psi(y) - \psi(x)$ can be replicated (Q^*) perfectly (CRR) by $b(x) + h(x)(y-x)$. Now $b = 0$ by taking expectations, so that the WEP holds on the support of Q^*, and therefore everywhere. Application: Hobson Klimmek.

Basic facts

Let $S(x)$ the support of $Q(x,.)$.Assume the WEP and $\forall x, x \in S(x)$.

Lemma

On $S(x)$, $y \to \psi(y) + f(x,y)$ is affine. In particular $\psi_S(x)$ is fully determined by its values at any 2 points.

Corollary

For distincts x, x', $\sharp S(x) \cap S(x') \leq 2$.

NB: if all the sets $S(x)$ are disjoints, then $M(\mu, \nu)$ is a singleton.

The point of interest is the combinatorics of the sets $\sharp S(x) \cap S(x')$

Denny's (non martingale) characterization

Theorem

Q is extremal in $\Pi(\mu, \nu)$ iff

- *$supp(Q) = \{(x, f(x)\} \cup \{(g(y), f(y)\}$ for 2 functions f, g*
- *for any n, $(g.f)^n$ has no fixed point*

Remark: Dom(f) or Dom(g) can be empty.

Denny's (and Letac) cycles

- The main idea in Denny's theorem is that it is possible to perturbate Q along a cycle.
- What about the martingale property? It will not be preserved by such a perturbation.

A martingale perturbation

Assume in the 3 points support case that $\sharp S(x) \cap S(x') = 2$, $\sharp S(x) \cap S(x'') = 2$, $\sharp S(x') \cap S(x') = 1$.

Then we can build a martingale perturbation.

A candidate cycle like property

Consider we start from a given $x \in X$. Set:

1. $\Psi_1 = S(x)$, $T_1 = \{x\}$
2. By recurrence, let $T_{n+1} = \{y \notin T_n \,/\, S(y) \cap \Psi_n \neq \emptyset\}$ and set $\Psi_{n+1} = \Psi_n \cup_{T_{n+1}} S(y)$, and for $z \in T_{n+1}$, $\Psi^*_{n+1}(z) = \Psi_n \cup_{T_{n+1} \setminus z} S(y)$ for $n \geq 1$.

Our sufficient condition read, in step 2 above:

$$\forall z \in T_{n+1}, \sharp(S(z) \cap \Psi^*_{n+1}(z)) \leq 2$$

A martingale cycle would be $z \in T_{n+1}, \sharp(S(z) \cap \Psi^*_{n+1}(z)) \geq 3$.

Conclusion

- WEP and sequential WEP from Douglas theorem
- Solving the WEP via the Poisson equation
- A candidate martingale cycle property
- Many questions remain!

Thank you for your attention !

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